Research and Application for the Regularity of Distribution about Curvature Radius and Curvature Center of the Linkage Point Track in a Link Mechanism

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Abstract: *It is illustrated that there exists an inflection circle on the linkage rigid body by the principle of relative motion. Confirmed methods of the inflection circle curvature radius and curvature center of the point track on the linkage rigid body are given in the case of the different contact type of move instantaneous center line and static in-stantaneous center line. The regularity of distribution of curvature radius and curvature center of the point track is researched. The identification methods called determination parameters and auxiliary vertical line of the diameter and direction of the inflection circle in the four bar mechanism are pointed out. A design method of the crane hoist-ing mechanism is discussed in the end of this paper.*

Key words: link mechanism; linkage point track; instantaneous center line; inflection circle; determination pa-rameters and auxiliary vertical line; curvature radius; curvature center; distribution regularity; the crane hoisting mechanism

1   Introduction

By the knowledge of mechanical principles we know planar linkages usually do generally planar motion. At every instant of time a linkage rigid body usually has a zero speed point. It is called the instantaneous cen-ter of velocity. For the four bar mechanism it is the intersection of two frame connecting rods. In the link-age motion process it draws a static instantaneous center line in the static coordinate system and draws a move instantaneous center line in the moving coordi-nate system fixed in the linkage body. The move in-stantaneous center line fixed in the linkage body is a pure roll about the static instantaneous center line fixed in the frame without slipping. The pure rolling constraint makes the linkage body be a single degree

system with the instantaneous moving distance s as a generalized coordinate. Giving the shape of move in-stantaneous center line and static instantaneous center line and the position of instantaneous contact point in the linkage body the point track curvature radius and curvature center will be determined. This paper stud-ies problems such as determining inflection circle curvature radius and curvature center of the point track on the linkage rigid body and the regularity of distribution of curvature radius and curvature center of the point track in an instant time. It can guide the de-sign of the crane hoisting mechanism six bar dwell mechanism approximate constant transmission ratio four bar mechanism and the drawing to the four bar linkage point track [1~14] . It also lays the necessary foundation for the system research of the stationary curvature point curve and the stationary curvature center curve.

Therefore starting from the basic knowledge of mechanics this paper builds the relationship between the instantaneous moving speed ds/dt with the curvature radius of static and move instantaneous center line and the linkage angular velocity. This relationship involves different contact types of move instantaneous center line and static instantaneous center line. Using the theory of relative motion the calculation formula of the acceleration of the instantaneous center and the diameter of the inflection circle in the linkage body are derived. Using the theory of differential geometry then the general formula of curvature radius of each point track on the linkage rigid body is given. This general formula depends on the inflection circle diameter and the point polar coordinate. Three graphic approaches are elaborated for the inflection circle diameter and the track curvature center. Through research the distribution regularity of the track curvature center and the change regularity of the track curvature radius are obtained. Then the new access approaches of the diameter and direction of the linkage inflection circle in the four bar mechanism is pointed out by the mechanism dimension.

The identification methods of the track curvature center and the inflection circle in the four bar mechanism are asupplementation to the Reference 15 .

As to the application of the theory and method this paper gives a design method of the crane hoisting mechanism.

2 Relationship of the linkage angular velocity and the acceleration of the instantaneous center

We abbreviate the instantaneous center line as a centrode. For a given mechanism the static centrode and the move centrode can be determined by a graphic method or analytic method. With the moving of the mechanism the instantaneous center of velocity P moves along the centrode goes a moving distance s varying with time t and moves out an instantaneous speed ds/dt. The move centrode pure rolling about the static centrode is tangent at P and has three

probable contact types. Another knowing the curvature radius of static and move centrode the relationship of the linkage angular velocity ω and the acceleration $\boldsymbol{a}_p$ of the instantaneous center on the linkage can be obtained as the following.
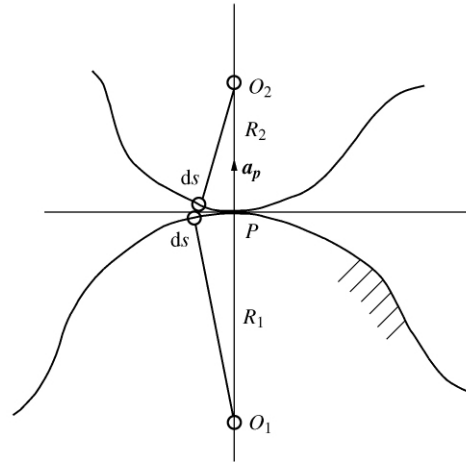
2. 1 Static and move centrodes are convex

Figure 1   Static and move centrodes are convex

As shown in Figure 1 the static and move centrodes are convex in the instantaneous center of velocity P. The curvature center of the static centrode is O_1 curvature radius is R_1. The curvature center of the move centrode is O_2 curvature radius is R_2. P moves along the static centrode for absolute motion and moves along the move centrode for relative motion. The move centrode around the static centrode do as an implicated roll. The linkage angular velocity ω is the sum of the derivations of the angle of the move centrode related to the instantaneous tangent line and instantaneous tangent line related to the static centrode to time idest:

$$\omega = \frac{ds}{dt}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad (1)$$

The absolute and relative tangential acceleration of the instantaneous center P is equal; absolute normal acceleration equals to $\left(\frac{ds}{dt}\right)^2 \frac{1}{R_1}$ pointing from dot P to dot O_1 relative normal acceleration equals to $(ds/dt)^2/R_2$ pointing from dot P to dot O_2; Geliaoli acceleration equals to $2\left(\frac{ds}{dt}\right)^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ also pointing from dot P to dot O_1. Using the theory of relative mo-

tion the mode calculation formula of implicated ac-celeration a_p of the instantaneous center P on the link-age rigid body can be derived as

$$a_p = D\omega^2 \qquad (2)$$

Where

$$\frac{1}{D} = \frac{1}{R_1} + \frac{1}{R_2} \qquad (3)$$

Its direction points from dot P to dot O_2. Its value is independent with the linkage angular acceleration ε.

While the static centrode is a straight line R_1 is infi-nite $1/R_1$ equals to 0.

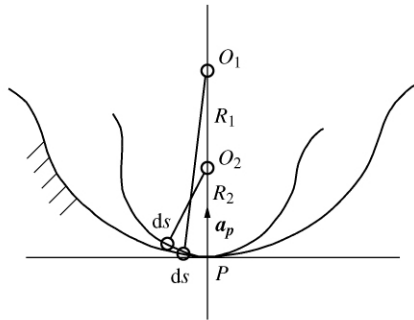## 2. 2 Move centrode convex and static centrode concave

Figure 2   Move centrode convex and static centrode concave

As shown in Figure 2   the move centrode is convex and the static centrode is concave in the instantaneous center of velocity P. The curvature center of the static centrode is O_1 curvature radius is R_1；the curvature center of the move centrode is O_2 curvature radius is R_2. Similarly we can give out the linkage angular velocity ω as

$$\omega = \frac{ds}{dt}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \qquad (4)$$

The mode calculation formula of implicated accelera-tion a_p of the instantaneous center P on the linkage rigid body also is Formula (2)

but where

$$\frac{1}{D} = \frac{1}{R_2} - \frac{1}{R_1} \qquad (5)$$

Its direction points from dot P to dot O_2. Its value is independent with the linkage angular acceleration ε too.

2. 3 Move centrode concave and static centrode

convex

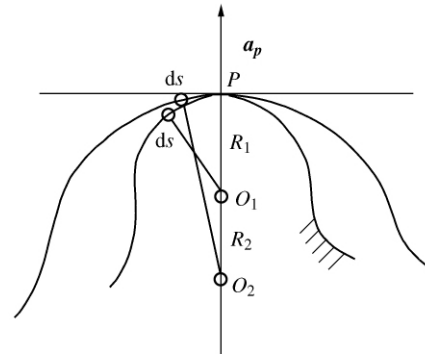Figure 3   Move centrode concave and static centrode convex

As shown in Figure 3   the move centrode is concave and the static centrode is convex in the instantaneous center of velocity P. The curvature center of static centrode is O_1 curvature radius is R_1；the curvature center of the move centrode is O_2 curvature radius is R_2. Similarly we give out the linkage angular velocity ω as

$$\omega = \frac{ds}{dt}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (6)$$

The mode calculation formula of implicated accelera-tion a_p of the instantaneous center P on the linkage rigid body also is Formula (2)

but where

$$\frac{1}{D} = \frac{1}{R_1} - \frac{1}{R_2} \qquad (7)$$

Along an instantaneous normal line its direction points from dot P to the move centrode side. Its value is independent with the linkage angular acceleration ε. While the move centrode is a straight line R_2 is infinite $1/R_2$ equals to 0.

3   Inflection circle 、instantaneous center of acceleration and general formula of cur-vature radius of each point track on the linkage rigid body

3. 1   Inflection circle and instantaneous center of acceleration

On the linkage rigid body   we define a circle located in the move centrode side   whoes tangent is at P and diameter equals to D as inflection circle as shown in
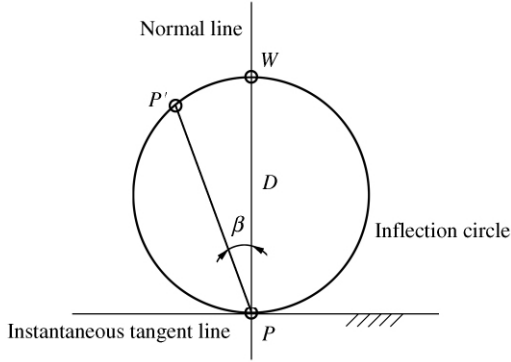
Figure 4.

Figure 4   Inflection circle and instantaneous
center of acceleration

The intersection W of the inflection circle with instantaneous normal line is defined as the inflection pole.

The normal acceleration of the point on the inflection circle is equal to the difference of the projection of instantaneous acceleration a_p on a track normal line with the product of the projection of PW and square of the linkage angular velocity ω; its result is zero. That is the acceleration and the velocity of the point on the inflection circle are in the same direction. By knowledge of mechanics [1]   curvature radius of the point track on the inflection circle is infinite. The instantaneous center of acceleration P' is also located on the inflection circle. Not difficult to prove the tangent of the angle β of PP' with PW equal to the ratio of linkage angular acceleration ε to the square of linkage angular velocity ω:

$$\tan\beta = \varepsilon/\omega^2 \qquad (8)$$

When the linkage angular acceleration ε is zero the instantaneous center of acceleration P' coincided with inflection pole W. When the linkage angular velocity ω is zero the instantaneous center of acceleration P' coincided with instantaneous center of velocity P.

The speed of inflection pole W equals to instantaneous speed ds/dt or product of the inflection circle diameter with the linkage angular velocity $D\omega$.

3. 2   Formula of curvature radius of the linkage point track

Knowing the contact type and the curvature radius of the static and move centrode the inflection circle the

curvature radius ρ and curvature center C^* of the track of every point C on the linkage rigid body will be determined.

The curvature center C^* is an inevitable locus on the connection of the point C with the instantaneous center P or the track normal line. We take the instantaneous normal line as the horizontal axis and the instantaneous tangent line as longitudinal axis and set instant natural coordinates xPy. The Px axis points to the move centrode side. The polar distance of the point C is r; polar angle is φ as shown in Figure 5.
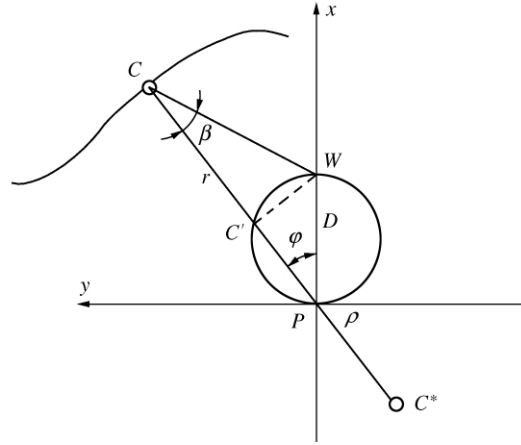

Figure 5   Polar coordinate and track curvature
center of linkage point C

We set the projection of the connection CW of the point C with the inflection pole W on the polar radius PC as CC' and its angle with the polar radius as β as shown in Figure 5. In order to export the calculation formula of curvature radius ρ of the track of linkage point C we consider the case that the move centrode pure roll about the static centrode with positive uniform angular velocity ω namely the instantaneous center of acceleration P' coincidence with the inflection pole W. By knowledge of differential geometry [13]   knowing the velocity V_C and the acceleration a_c of the linkage point C the mode calculation formula of its curvature radius ρ is

$$|\rho| = \frac{|V_C|^3}{|V_C \times A_C|} \qquad (9)$$

as shown in Figure 5   the projection CC' equal to

$$CC' = CW\sin\left(\frac{\pi}{2} + \beta\right) = r - D\cos\varphi \qquad (10)$$

so

$$|\rho| = \left| \frac{(r\omega)^3}{r\omega \cdot (r - D\cos\varphi)\omega^2} \right| \qquad (11)$$

see the Reference 12, written by the author, take simplification to Formula (11), remove the absolute value symbol, get an extended formula:

$$\rho = \frac{r^2}{r - D\cos\varphi} \qquad (12)$$

By comprehensive analysis visible, apparently this is the general formula of curvature radius ρ of each point track on the linkage rigid body. It depends on the inflection circle diameter D and the point polar coordinate. The calculation result is an algebraic value and equivalent to the Euler-Safali equation. When linkage point C locates at the inflection circle, r equal to $D\cos\varphi$, the denominator is zero, the curvature radius ρ is infinite and the track curvature center C^* locates at infinity. When point C is inside the inflection circle, the denominator less than zero, the calculation result of ρ is negative and the track curvature center

C^* is in the PC ray. When point C is outside the inflection circle but also located in the move centrode side, the denominator greater than zero, the calculation result of ρ is positive and the track curvature center C^* is in the CP ray. When point C locates in the static centrode side, the denominator also greater than zero, the calculation result of ρ is positive, and the track curvature center C^* is in the CP segment.

4 Graphic method of inflection circle diameter and track curvature center of linkage point

4.1 Graphic method of inflection circle diameter

Given the contact type and the curvature center of static and move centrodes, the length of the inflection circle diameter D can be obtained using the method shown in figure 6. In this Figure, the instantaneous center of velocity is P; the curvature center of the static centrode is O_1 and the curvature enter of move centrode is O_2.
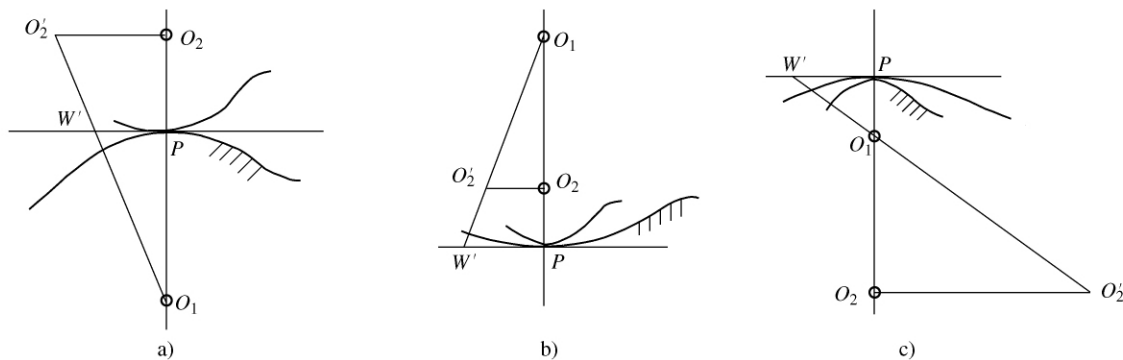

Figure 6   Graphic method of inflection circle diameter

Figure 6a) is the case that the static and move centrodes are convex; Figure 6b) is the case that the move centrode is convex and static centrode concave; Figure 6c) is the case that the move centrode is concave and static centrode convex. Make O_2O_2' perpendicular to PO_2, and have the same length with PO_2. Connect O_1O_2' intersect with instantaneous tangent line at W' and call it as an auxiliary point. Thus, the length of PW' is equal to the inflection circle diameter D. By use of the similarity of right triangle $O_1O_2O_2'$ and right triangle O_1PW', combined with Formulas (3)、(5)、(7), it is not difficult to prove the correctness of the

approach.

4.2 Graphic method of track curvature center of linkage point C

4.2.1 Using the auxiliary circle

As shown in Figure 7, we call the circle using PW' with the diameter as the auxiliary circle. Make CC' perpendicular to PC, and have the same length with PC. Then through P make a perpendicular line of PC, intersect with auxiliary circle at P'. Connecting $P'C'$ intersect with polar radius PC at C^*, we get the track curvature center. Set the projection of P' on CC' as C'', by use of the similarity of right triangle $C^* CC'$ and

right triangle $P'C''C'$ combined with Formula(12) it is not difficult to prove the correctness of the approach.

Figure 7　Using the auxiliary circle to track curvature center C^*

4. 2. 2　Using the inflection circle

As shown in Figure 8 through instantaneous center P make a perpendicular line of polar radius PC intersect with the connection of the inflection pole W with the point C at C'. Then through C' make a parallel line of PW intersect with PC at C^* we get the track curvature center. Set the projection of W on PC as C'' by use of the similarity of right triangle PCC' and right triangle WCC'' and the relationship of side and angle in right triangle $PC'C^*$ combined with Formula(12)

it is not difficult to prove the correctness of the approach.

Figure 8　Using the inflection circle to
track curvature center C^*

4. 2. 3　Using the curvature center of static and move centrode

As shown in Figure 9 the curvature center of the static centrode is O_1 and the curvature center of the move centrode is O_2. Through instantaneous center P make a perpendicular line of polar radius PC intersect with the connection of the point C with O_2 at C'. Then make the connection of C' with O_1 intersect with PC at C^* we get the track curvature center. This graphic approach has been introduced in the classical References 6 10 . So we omit the proof process.

Figure 9　Using the curvature center of static and move
centrode to track curvature center C^*

5　Regularity of distribution about the curvature center and curvature radius of the linkage point track

According to the approach using the auxiliary circle to track curvature center visible in every case the track curvature center of point O_2 will be O_1. The track curvature center of each point on the instantaneous tangent line will be instantaneous center P.

By a series of mathematical operations on Formula (12) using a ratio theorem we obtain

$$\frac{r}{\cos\varphi} + \frac{\rho - r}{\cos\varphi} = \frac{1}{D} \cdot \frac{r}{\cos\varphi} \cdot \frac{\rho - r}{\cos\varphi} \quad (13)$$

Fomula we know that when ratio $\dfrac{r}{\cos\varphi}$ is a constant

value ratio $\dfrac{\rho - r}{\cos\varphi}$ will be another constant value.

Then the track curvature center C_i^* of the point C_i on the circle tangent with the instantaneous tangent line at P will be on another circle tangent with the instantaneous tangent line at P.

Along the negative direction of the x axis set PW'' equal to the inflection circle diameter D. We call the circle using PW'' with the diameter as the limit circle.

Apparently the limit circle images the inflection circle on each other. According to the approach using the auxiliary circle to track the curvature center visible along the positive direction of the x axis take the linkage point C from P to W its track curvature center C^* will change from P to infinity along the positive direction of the x axis. Then take the linkage point C from W to positive infinity its track curvature center C^* will change from negative infinity to W''. Then take the linkage point C from negative infinity to P; its track curvature center C^* will change from W'' to P. Combined with Formula（13） when linkage point C is inside the inflection circle the track curvature center C^* will be distributed in the whole region of the move centrode side. When linkage point C locates at the inflection circle the track curvature center C^* will be distributed in infinity. When linkage point C is outside the inflection circle but also located in the move centrode side the track curvature center C^* will be distributed in the static centrode side but also outside the limit circle. When linkage point C locates in the whole region of the static centrode side the track curvature center C^* will be distributed inside the limit circle. When linkage point C locates in infinity the track curvature center C^* will be distributed at the limit circle.

Figure 10 is the diagram of the distribution about curvature center of the linkage point track.
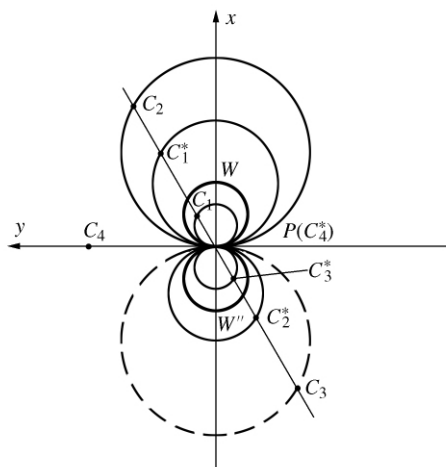

Figure 10   Diagram of the distribution about curvature
center of the linkage point track

In addition by Formula（12）the coordinate of the linkage point C on the x axis changes from 0 to $D/2$ and the absolute value of track curvature radius will increase from 0 to $D/2$ smoothly. The coordinate changes from $D/2$ to D the absolute value of track curvature radius will increase from $D/2$ to infinity rapidly. The coordinate changes from D to $1.5D$ and track curvature radius will decrease from infinity to $4.5D$ rapidly. As the coordinate changes from $1.5D$ to $2D$ track curvature radius will decrease from $4.5D$ to $4D$ smoothly and reach the minimum value. As the coordinate changes from $2D$ to $3D$ track curvature radius will increase from $4D$ to $4.5D$ smoothly. As the coordinate changes from $3D$ to infinity track curvature radius will increase from $4.5D$ to infinity smoothly.

As the coordinate changes from negative infinity to $-D$ track curvature radius will decrease from infinity to $D/2$ smoothly. As the coordinate changes from $-D$ to 0 track curvature radius will decrease from $D/2$ to 0 smoothly.

By Formula（13） the track curvature radius variable regularity of the point on the straight line through P is similar with that on the instantaneous normal line. If it presents a positive or negative angle φ with the instantaneous normal line its curvature radius function is shortened by $\cos\varphi$ multiple.

6   The identification of the inflection circle in a four bar mechanism and the design of the crane hoisting mechanism

As shown in Figure 11 for an instant time of the four bar mechanism knowing two move hinges A、B on the linkage rigid body and the corresponding track curvature center（fixed hinges）A^*、B^* the instantaneous center of velocity P of the linkage AB will be the intersection of two frame connecting rods AA^* and BB^*. Two polar radius r_1、r_2 and their cross angle δ can be determined also.

By the general calculation Formula（12）of the track curvature radius We get two simultaneous equations
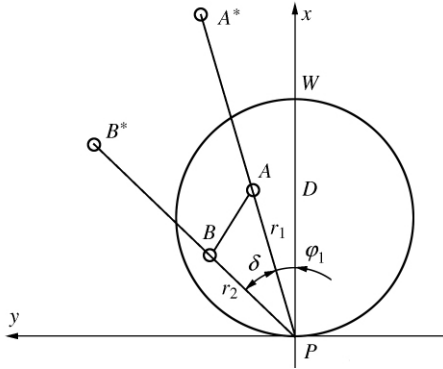
Figure 11   Identification of the diameter and direction
of the inflection circle by two move hinges
on a linkage body and two fixed hinges
of the mechanism

$$\begin{cases} \rho_1 = \dfrac{r_1^2}{r_1 - D\cos\varphi_1} \\ \rho_2 = \dfrac{r_2^2}{r_2 - D\cos(\varphi_1 + \delta)} \end{cases} \quad (14)$$

Separate polar angle φ_1 of PA and inflection circle diameter D from it the shape and direction of the inflection circle on the linkage rigid body will be determined singly. We call this identification method as the determination parameters method.

As illustrated in Figure 7 prior knowing the linkage point C the instantaneous center P and the track curvature center C^* then we can decide point C' and point P'. Through P' make $P'C''$ perpendicular to PP' the auxiliary point W' shall locate on it. We call this perpendicular line $P'C''$ as the auxiliary vertical line.

For the four bar mechanism shown in Figure 11 using A、P and A^* take an auxiliary vertical line then using B、P and B^* take another auxiliary vertical line they shall intersect at auxiliary point W'. So the straight line where $P W'$ will be the instantaneous tangent line the length of PW' will be the inflection circle diameter D. We call this graphic identification method as the auxiliary vertical line method.

In addition take the inflection pole W as the linkage point of a four bar mechanism; its track will close to the straight line tangent with the inflection circle at W. So we give out a design method of the crane hoisting mechanism as following.

As shown in Figure 12 in order to ensure the jib pulley center track close to a level straight line we take the positive direction of the instantaneous normal line Px downward verticaly to the ground. Arbitrarily take the inflection circle diameter D according to the actual situation. Then take the inflection pole W as the jib pulley center and properly select the arm position line and the position of two move hinges A、B. According to the graphic method using the inflection circle to track the curvature center obtain two corresponding track curvature centers A^*、B^* as fixed hinges get out a hinge four bar mechanism $B^* BAA^* W$ to complete the design of the crane hoisting mechanism.
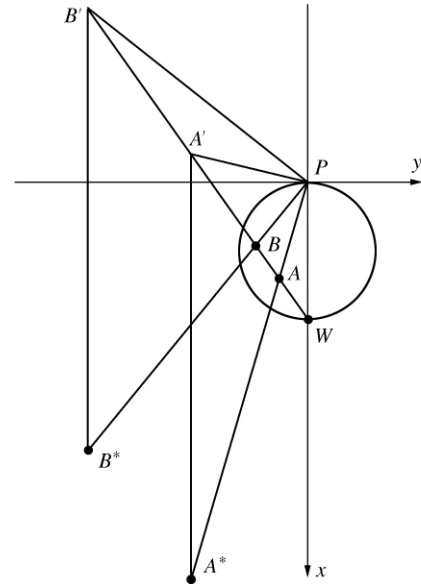

Figure 12   Design of the crane hoisting mechanism

Take $D = 100$ cm $\angle PWA = 28°$ $\angle WPA = 30°$ $\angle WPB = 43°$ design out the crane hoisting mechanism and the jib pulley center track as shown in Figure 13.
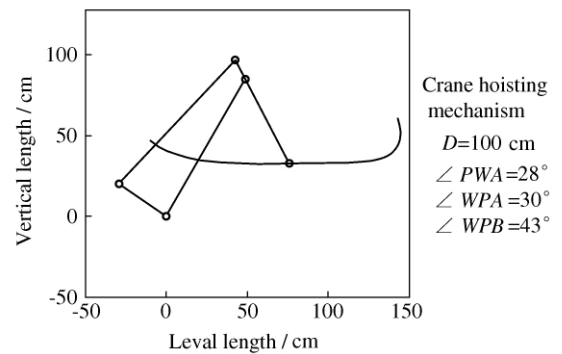

Figure 13   Design example

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